PARAMETRIC FAST FOURIER TRANSFORM

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Fast Fourier Transform history overview

- FFT: Cooley-Tukey (1965). Originaly it was invented by Carl Friedrich Gauss (1805).
- The Fourier matrix decomposition: Pease (1968)
- Prime factor algorithm: Good (1958)
- split-radix FFT: Yavne (1968)
- Small FFTs algorithms: Winograd (1978)
- Parametric representation of indices: Sverdlik (1984)
- Notes on the FFT: Burrus (1997)

Fast Fourier Transform modern history

- A methodology for designing, modifying and implementing Fourier transform algorithms on various architectures: Johnson J., Johnson R. W., Rodriguez D., Tolimieri R. (1990)
- FFTW: Frigo, Johnson (2005)
- SPIRAL: Püschel M., Moura J. M. F., Johnson J., Padua D., Veloso M., Singer B., Xiong J., Franchetti F., Gačić A., Voronenko Y., Chen K., Johnson R. W., Rizzolo N. (2005)
- Parametric Fast Fourier Transform: Prosekov, Malozemov (2008)

Fast Fourier Transform applications

- Fast large-integers and polynomial multiplication
- Efficient matrix-vector multiplication for Toeplitz, circulant and other structured matrices
- Filtering (convolution) algorithms
- Solving difference equations
- Fast Discrete Cosine and Sine transforms (e. g. fast DCT used JPEG and MP3/MPEG encoding/decoding)
- Convolution Neural Network (CNN)

Parametric coding of indices

Consider the Fourier matrix F_N with elements

$$F_N[k,j] = \omega_N^{kj}\,, \quad k,j \in 0:N-1\,, \quad \omega_N = \exp(2\pi i/N)\,.$$

Let $N = n_1 n_2 \cdots n_s$. Suppose that, for each $\nu \in 1:s$, there exits numbers p_ν, q_ν (parameters) coprime to n_ν and numbers D_ν, G_ν (bases) such that

$$j = \left\langle \sum_{\nu=1}^{s} j_\nu\, p_\nu D_\nu \right\rangle_N, \qquad j_\nu \in 0:n_\nu - 1\,,$$

$$k = \left\langle \sum_{\mu=1}^{s} k_\nu\, q_\nu G_\nu \right\rangle_N, \qquad k_\nu \in 0:n_\nu - 1\,.$$

Important notations

Let

$$N = n_1 n_2 \cdots n_s\,;$$

$$\Delta_\nu = n_1 n_2 \cdots n_{\nu-1} \qquad \text{for } \nu \in 2:s+1, \quad \Delta_1 = 1\,;$$

$$N_\nu = n_{\nu+1} n_{\nu+2} \cdots n_s \quad \text{for } \nu \in 0:s-1, \quad N_s = 1\,.$$

Obviously, $\Delta_\nu\, n_\nu\, N_\nu = N$ for all $\nu \in 1:s$. Any number $j \in 0:N-1$ can be uniquely represented in the form

$$j = \sum_{\nu=1}^{s} j_\nu\, \Delta_\nu\,, \qquad j_\nu \in 0:n_\nu - 1\,.$$

$j = (j_s, j_{s-1}, \ldots, j_1)_{n_s, n_{s-1} \ldots, n_1}$ — mixed code of the number j.

We use Kronecker multiplication of matrices: $A \otimes B = \big[a_{ij}\, B\big]$

Parametric Fast Fourier Transform

Choice of bases: $D_\nu = \Delta_\nu$ and $G_\nu = N_\nu$.

THEOREM. For any parametric vector $p = (p_1, p_2, \ldots, p_s)$, the Fourier matrix F_N admits the representation

$$F_N = \left(Rev^{(q_1,\ldots,q_s)}_{n_1,\ldots,n_s}\right)^T \left(\prod_{\nu=1}^{s} \left(I_{N_\nu} \otimes Twid^{(p_1,\ldots,p_{\nu-1},q_\nu)}_{n_1,\ldots,n_{\nu-1},n_\nu}\right) \times \times \left(I_{N_\nu} \otimes F_{n_\nu} \otimes I_{\Delta_\nu}\right)\right) Mix^{(p_1,\ldots,p_s)}_{n_1,\ldots,n_s} ,$$

where $q = (q_1, q_2, \ldots, q_s)$ is the adjoint parametric vector with elements are defined by the condition $\langle q_\nu\, p_\nu \rangle_{n_\nu} = 1$ for $\nu \in 1 : s$.

Diagonal parametric twiddle matrix

$$Twid^{(p_1,\ldots,p_{\nu-1},p_\nu)}_{n_1,\ldots,n_{\nu-1},n_\nu}[j,j] = \omega_{\Delta_{\nu+1}}^{j_\nu\, p_\nu \sum_{\alpha=1}^{\nu-1} j_\alpha\, p_\alpha \Delta_\alpha}, \quad \nu \in 2:s\,.$$

Here, $j = (j_\nu, j_{\nu-1}, \ldots, j_1)_{n_\nu, n_{\nu-1},\ldots,n_1}$. $Twid^{(p_1)}_{n_1} := I_{n_1}$.

Example: $N = 2 \cdot 3 \cdot 4$, $p = (3,2,3)$ and $q = (1,2,3)$

Number of nontrivial elements (different from ± 1 and $\pm i$):

$Twid^{(1,1)}_{2,3}$ — 2 elements, $\quad Twid^{(3,2)}_{2,3} = I_6$,

$Twid^{(1,1,1)}_{2,3,4}$ — 12 elements, $\quad Twid^{(3,2,3)}_{2,3,4}$ — 6 elements.

Parametric permutations

Let $j = (j_s, \ldots, j_1)_{n_s,\ldots,n_1}$. We introduce permutations

$$\operatorname{mix}^{(p_1,p_2,\ldots,p_s)}_{n_1,n_2,\ldots,n_s}(j) = \left\langle \sum_{\nu=1}^{s} j_\nu\, p_\nu \Delta_\nu \right\rangle_N,$$

$$\operatorname{rev}^{(p_1,p_2,\ldots,p_s)}_{n_1,n_2,\ldots,n_s}(j) = \left\langle \sum_{\nu=1}^{s} j_\nu\, p_\nu N_\nu \right\rangle_N.$$

Parametric permutation matrices $Mix^{(p_1,\ldots,p_s)}_{n_1,\ldots,n_s}$ and $Rev^{(p_1,\ldots,p_s)}_{n_1,\ldots,n_s}$ are defined in a standard way. For example

$$Rev^{(p_1,\ldots,p_s)}_{n_1,\ldots,n_s}[i,j] = \begin{cases} 1, & \text{if } j = \operatorname{rev}^{(p_1,\ldots,p_s)}_{n_1,\ldots,n_s}(i), \\ 0 & \text{otherwise.} \end{cases}$$

Parametric prime factor method

Consider most strong example. Let the factors n_ν be pairwise coprime. Then

$$Mix_{n_1,n_2,\dots,n_\nu}^{(p_1N_1,p_2N_2,\dots,p_\nu N_\nu)} = Rev_{n_1,n_2,\dots,n_\nu}^{(p_1\Delta_1,p_2\Delta_2,\dots,p_\nu\Delta_\nu)} = \\ =: Perm_{n_1,n_2,\dots,n_\nu}^{(p_1,p_2,\dots,p_\nu)} .$$

In this case $Twid_{n_1,\dots,n_{\nu-1},n_\nu}^{(p_1N_1,\dots,p_{\nu-1}N_{\nu-1},p_\nu\Delta_\nu)} = I_{\Delta_{\nu+1}}$.

The theorem implies factorization (like Good's PFA)

$$F_N = \left(Perm_{n_1,\dots,n_s}^{(q_1,\dots,q_s)}\right)^T \left(F_{n_s} \otimes \cdots \otimes F_{n_1}\right) Perm_{n_1,\dots,n_s}^{(p_1,\dots,p_s)} ,$$

where $\langle q_\nu \Delta_\nu \, p_\nu N_\nu \rangle_{n_\nu} = 1$ for $\nu \in 1 : s$.

Decomposition of parametric permutation matrices

THEOREM. For $s \geqslant 2$ the following decompositions hold:

$$Mix_{n_1,n_2,\dots,n_s}^{(p_1,p_2,\dots,p_s)} = \prod_{\nu=0}^{s-1} \left(Mix_{n_{s-\nu},N_{s-\nu}}^{(p_{s-\nu},1)} \otimes I_{\Delta_{s-\nu}} \right),$$

$$Rev_{n_1,n_2,\dots,n_s}^{(p_1,p_2,\dots,p_s)} = \prod_{\nu=1}^{s} \left(I_{N_\nu} \otimes Rev_{\Delta_\nu,n_\nu}^{(1,p_\nu)} \right).$$

Parametric permutation are connected by equation

$$Rev_{n_1,\dots,n_s}^{(p_1,\dots,p_s)} = Rev_{n_1,\dots,n_s}^{(1,\dots,1)} Mix_{n_s,\dots,n_1}^{(p_s,\dots,p_1)}.$$

Parallel-Vector Computations

$I_{\Delta_\nu} \otimes F_{n_\nu} \otimes I_{N_\nu}$

Example: Let $X = (I_2 \otimes F_2 \otimes I_2)\, x$

$$I_2 \otimes F_2 \otimes I_2 = \begin{bmatrix} I_2 & I_2 & 0_2 & 0_2 \\ I_2 & -I_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & I_2 & I_2 \\ 0_2 & 0_2 & I_2 & -I_2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} X_0 \\ X_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}, \qquad \begin{bmatrix} X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} - \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}, \qquad \begin{bmatrix} X_6 \\ X_7 \end{bmatrix} = \begin{bmatrix} x_6 \\ x_7 \end{bmatrix} - \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$$

2D DFT

Let $x(j_1 + j_2\, m_1) = x(j_1, j_2)$ and

$$X = \left(\bar{F}_{m_2} \otimes \bar{F}_{m_1}\right) x\,,$$

where $\bar{F}_{m_\mu}[k, j] = \omega_{m_\mu}^{-kj}$, $k, j \in 0 : m_\mu - 1$, $\mu \in 1 : 2$. Then

$$X(k_1, k_2) = X(k_1 + k_2\, m_1).$$

or maxtrix form

$$X = \bar{F}_{m_1}\, x\, \bar{F}_{m_2}$$

NOP (Flatten) is a layer in neural networks that reshapes the input data into a one-dimensional array without changing its content.

References

1. Malozemov V. N., Prosekov O. V. Parametric Versions of the Fast Fourier Transform // Doklady Mathematics. 2008. Vol. 78. No. 1. P. 576–578. 10.1134/S1064562408040285.
2. Malozemov V. N., Prosekov O. V. Parametric Versions of Fast Fourier Transform // LAP LAMBERT Academic Publishing, 2010, 124 pages, ISBN 978-3-8433-0429-0, www.morebooks.de

Implementations

1. ОАО ОКЕАНПРИБОР (Морфизприбор): ГАК МГК-400ЭМ-03 МГК-400ЭМ (2007)
2. Virage Logic (ARC): Real-Valued FFT Implementation (2009)
3. Samsung: 2-dimensional Fast Fourier transform for sizes Kx8 (2020)

Thank you!

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