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Радиальная непрерывность, сходимость по Дойчу и приложения к экспоненциальным суммам и рациональным функциям

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Below:

$X = (X, \|\cdot\|)$ is a normed linear space over $\mathbb{R}$,

$\mathring{B}(x,r) = \{y \in X \mid \|y - x\| < r\}$ is the open ball of radius r with center at x;

$B(x,r) = \{y \in X \mid \|y - x\| \leqslant r\}$ is the closed ball;

$S(x,r) = \{y \in X \mid \|y - x\| = r\}$ is the sphere.

For brevity, $B := B(0,1)$ is the unit ball, $\mathring{B} := \mathring{B}(0,1)$ is the open unit ball, $S = S(0,1)$ is the unit sphere; S^* is the dual unit sphere.

The *distance* from a point $x \in X$ to a nonempty set $M \subset X$ is defined by $\rho(x,M) := \inf_{y \in M} \|y - x\|$. The set of *nearest points* for a point $x \in X$ from a set M is defined by

$$P_M x := \big\{y \in M \mid \rho(x,M) = \|y - x\|\big\}.$$

A set M is an existence set (a proximinal set) if $P_M x \neq \varnothing$ for each $x \in X$.

1 Radial continuity

Definition 1.1. The metric projection P_M is called IRL-*continuous* (*inner radially lower continuous*) at a point x_0 if, for each $y_0 \in P_M x_0$ and each open W such that $W \cap P_M x_0 \neq \varnothing$, there exists a neighborhood U of x_0 such that $P_M x \cap W \neq \varnothing$ for each $x \in U \cap \{y_0 + \lambda(x_0 - y_0) \mid 0 \leqslant \lambda \leqslant 1\}$.

We note the following properties equivalent to the IRL-continuity of the operator P_M at a point x_0.

Lemma 1.A. *Let $M \subset X$, $x_0 \in X$, $P_M x_0 \neq \varnothing$. Then the following properties are equivalent:*

1) *the operator P_M is* IRL-*continuous at x_0;*

2) *for all $v_0, v_1 \in P_M x_0$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that $P_M x \cap B(v_1, \varepsilon) \neq \varnothing$ for all $x \in \{v_0 + \lambda(x_0 - v_0) \mid 1 - \delta < \lambda \leqslant 1\}$;*

3) *for each $v_0, v_1 \in P_M x_0$ and each sequence $(x_n) \subset \{v_0 + \lambda(x_0 - v_0) \mid 0 \leqslant \lambda \leqslant 1\}$, $x_n \to x_0$, there exists a point $v_n \in P_M x_n$ such that $v_n \to v_1$.*

Theorem 1.A (Brosowski and Deutsch)**.** *If, for some x, the set of nearest points $P_M x$ is convex, then the metric projection is* IRL-*continuous at x.*

A segment in a normed linear space X is defined as follows:

$$\begin{aligned}[\![x,y]\!] :=&\big\{z \in X \mid \min\{\varphi(x),\varphi(y)\} \leqslant \varphi(z) \leqslant \max\{\varphi(x),\varphi(y)\} \\ &\qquad\qquad\qquad\qquad \forall \varphi \in \operatorname{ext} S^*\big\} \\ =&\big\{z \mid \varphi(z) \in [\varphi(x),\varphi(y)] \quad \forall \varphi \in \operatorname{ext} S^*\big\},\end{aligned}$$

where $\operatorname{ext} S^*$ is the set of extreme points of the dual unit sphere S^* of X^*.

The relative interior of a segment $[\![x,y]\!]$ is defined as follows:

$$[\![x,y]\!]^0 :=\big\{z \in [\![x,y]\!] \mid \varphi(z) \in (\varphi(x),\varphi(y)) \ \ \forall \varphi \in \operatorname{ext} S^* \ : \varphi(x) \neq \varphi(y)\}. \tag{1}$$

Definition 1.2. A curve $k(t)$ is *monotone* if $f(k(t))$ is a monotone function of f for each extreme functional t of the dual unit sphere S^*.

Definition 1.3. A set $M \subset X$ is called *strictly monotone path-connected* if any two points x, y from M can be joined by a monotone curve $k(\,\cdot\,) \subset M$ which lies (except for the points x and y) in $[\![x,y]\!]^0$ (see (1)).

Definition 1.4. A set $M \subset X$ is a *sun* if each point $x \in X \setminus M$ is a solar point for M (i.e., x has at least one luminosity point in M). A set $M \subset X$ is a *strict sun* if each point $x \in X \setminus M$ is a strict solar point for M (i.e., if the set of nearest point $P_M x$ is nonempty and each nearest point for x is a luminosity point). A set $M \subset X$ is a *strict protosun* if each point $x \in X \setminus M$ is a strict protosolar point for M (i.e., if each nearest point from M for x is a luminosity point). Strict protosuns are precisely the sets satisfying the generalized Kolmogorov criterion for best approximants.

Theorem 1.B. *A monotone path-connected Chebyshev set is a sun.*

Theorem 1.C. *Each strictly monotone path-connected set is a strict protosun, i.e., if M is strictly monotone path-connected set, then, for all $x \in \operatorname{out} M := \{z \mid \rho(z, M) > 0\}$ and $y \in P_M(x)$, y is a luminosity point in M for x.*

The next theorem extends Theorem 1.A to sets with not necessarily convex-valued metric projection.

Theorem 1.1. *If, for some x, the set of nearest points $P_M x$ is strictly monotone path-connected, then the metric projection is* IRL*-continuous at x.*

Definition 1.5. Let $M \subset X$ and $x_0 \in X$. The metric projection P_M is called ORU-*continuous* (*outer radially upper continuous*) at a point x_0 if, for each $y_0 \in P_M x_0$ and each open $W \supset P_M x_0$, there exists a neighborhood U of x_0 such that $P_M x \subset W$ for each $x \in U \cap \{y_0 + \lambda(x_0 - y_0) \mid \lambda \geqslant 1\}$.

Lemma 1.B. *Let $M \subset X$, $x_0 \in X$, $P_M x_0 \neq \varnothing$. Consider the following properties:*

1) *the metric projection operator P_M is* ORU-*continuous at x_0;*

2) *for each $v_0 \in P_M x_0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sup_{v \in P_M x} \rho(v, P_M x_0) < \varepsilon$$

for each $x \in \{v_0 + \lambda(x_0 - v_0) \mid 1 \leqslant \lambda < 1 + \delta\}$;

3) *for each point $v_0 \in P_M x_0$ and each sequence $(x_n) \subset \{v_0 + \lambda(x_0 - v_0) \mid \lambda \geqslant 1\}$, $x_n \to x_0$,*

$$\sup_{v \in P_M x_n} \rho(v, P_M x_0) \to 0;$$

4) *for each point $v_0 \in P_M x_0$, each sequence $(x_n) \subset \{v_0 + \lambda(x_0 - v_0) \mid \lambda \geqslant 1\}$, $x_n \to x_0$, and each sequence (v_n), $v_n \in P_M x_n$,*

$$\rho(v_n, P_M x_0) \to 0;$$

5) *for each point* $v_0 \in P_M x_0$, *each sequence* $(x_n) \subset \{v_0 + \lambda(x_0 - v_0) \mid \lambda \geqslant 1\}$, $x_n \to x_0$, *and each sequence* (v_n), $v_n \in P_M x_n$ *and* $v_n \to v$,

$$v \in \overline{P_M x_0}.$$

Then 2)⇔3)⇔4)⇒5). *If* $P_M x_0$ *is compact, then* 4) ⇒1) *and the first four conditions are equivalent. If* M *is compact, then* 5)⇒1) *and the first five conditions are equivalent.*

Theorem 1.D (Brosowski and Deutsch)**.** *Let M be a strict protosun with compact-valued metric projection P_M. Then P_M is* ORU *continuous.*

Theorem 1.2. *Let M be a proximinal prox-strictly monotone path-connected set and the metric projection onto M has nonempty compact values. Then M is a strict sun with* IRL- *and* ORU-*continuous metric projection.*

2 Exponential sums

J. Rice and G. Meinardus seem to have been the first to study approximation by exponential sums. Their studied were continued, among others, by D. Braess, H. Werner, E. Schmidt, D. W. Kammler, Ch. Dunham, V. I. Berdyshev, R. Schaback, G. Nürnberger, I. G. Tsar'kov, A. R. Alimov, etc. (see, e.g., [1]).

Consider the following classical classes of one-dimensional exponential sums (here, $x \in \mathbb{R}$):

$$E_n^+ := \left\{ \sum_{j=1}^{n} \alpha_j e^{t_j x} \;\middle|\; \alpha_j \geqslant 0,\ t_j \in \mathbb{R} \right\}, \tag{2}$$

$$E_n^0 := \left\{ \sum_{j=1}^{n} \alpha_j e^{t_j x} \;\middle|\; \alpha_j, t_j \in \mathbb{R} \right\}, \tag{3}$$

$$E_n := \overline{E_n^0} \quad \text{(the set of extended exponential sums)}. \tag{4}$$

Remark 2.1. The set of exponential sums E_n^0 is a strict protosun of uniqueness in $C[a,b]$, the set exponential sums E_n^+ with nonnegative coefficients is a Chebyshev sun in $C[a,b]$, the set E_n is proximinal but (for $n \geqslant 2$) is not a uniqueness set, and the metric projection onto E_n, $n \geqslant 2$, may have local minima which are not global (see, e.g., [1]).

Theorem 2.1. *The metric projection onto the sets E_n^+ and E_n^0 of univariate exponential sums* (*see* (2) *and* (3)) *is both* ORU- *and* IRL-*continuous.*

Definition 2.1. Let Ω be a compact convex body in $\mathbb{R}^d$. A sequence of functions (f_n) from $C(\Omega)$ *converges uniformly inside* Ω to a function $f \in C(\Omega)$ if (f_n) converges uniformly to f on the set $\Omega^\delta := \{\mathbf{x} \in \Omega \mid \rho(\mathbf{x}, \operatorname{bd}\Omega) \geqslant \delta\}$ for each $\delta > 0$. This convergence is a particular case of the sequential Deutsch Δ-convergence.

Remark 2.2. The set E_n^+ and the set E_n (of one-dimensional extended exponential sums; see (2) and (4) above) are boundedly compact relative to the uniform convergence inside (a, b). As a result, the metric projection onto the set E_n, $n \geqslant 2$, of extended exponential sums (see (4)) cannot be both IRL- and ORU-continuous or even both IRL- and D-ORU-continuous, where D is the uniform convergence inside (a, b).

We consider the following sets of multivariate exponential sums:

$$\mathbf{E}_n^+ := \left\{ \sum_{1\leqslant j\leqslant n} a_j e^{(\boldsymbol{\lambda}_j,\mathbf{x})} \;\middle|\; \mathbf{x},\boldsymbol{\lambda}_j \in \mathbb{R}^d, \quad a_j \geqslant 0 \right\}, \tag{5}$$

$$\mathbf{E}_n := \left\{ \sum_{1\leqslant j\leqslant n} p_j(\mathbf{x}) e^{(\boldsymbol{\lambda}_j,\mathbf{x})} \;\middle|\; \mathbf{x},\boldsymbol{\lambda}_j \in \mathbb{R}^d, \quad \sum_{j=1}^{n} (1+\deg p_j(\mathbf{x})) \leqslant n \right\}, \tag{6}$$

where $p_j(\mathbf{x})$ is a real polynomial, $\mathbf{x} \in \mathbb{R}^d$. Here, the multiplied of $e^{(\boldsymbol{\lambda}_j,\mathbf{x})}$ by vectors a_j and $p_j(\mathbf{x})$ is defined coordinatewise.

Remark 2.3. The sets $\mathbf{E}_n$, $\mathbf{E}_n^+$ and $\mathbf{E}_n^- := -\mathbf{E}_n^+$ are boundedly Deutsch compact relative to the inner uniform convergence on Ω (see [1]). The set $\mathbf{E}_n^+$ is a monotone path-connected strict sun in $C(Q)$. Note that in general $\mathbf{E}_n^+$ for $d \geqslant 2$ is not a uniqueness set in $C(\Omega)$.

We also consider the following class of exponential sums:

$$\mathbf{E} = \{ a e^{(\boldsymbol{\lambda},\mathbf{x})} \mid \boldsymbol{\lambda} \in \mathbb{R}^d, \quad a \in \mathbb{R} \}, \quad \mathbf{x} \in \Omega, \tag{7}$$

where Ω is a compact convex body in $\mathbb{R}^d$, 0 is an interior point of Ω.

It is known that

$$\mathbf{E} \textit{ is a monotone path-connected strict sun in } C(\Omega). \tag{8}$$

We have the following result on radial continuity of the metric projection onto some classes of multivariate exponential sums. (Note that in general the metric projection here is discontinuous.)

Theorem 2.2. *Let Ω be a compact convex body in $\mathbb{R}^d$, $0 \in \operatorname{int}\Omega$, and D be the uniform convergence inside Ω. Then the metric projection onto the sets $\mathbf{E}_n^+$, $\mathbf{E}_n^-$, $\mathbf{E}$ of multivariate exponential sums (see (2) and (7)) is D-*ORU*-continuous.*

3 Generalized rational functions

Classical and generalized rational functions constitute a very important object of approximation theory, which evokes great theoretical and practical interest, especially in numerical mathematics. It is well known that the metric projection onto the set of classical rational functions

$$\mathscr{R}_{n,m} := \{r = p/q \in C[a,b]\}, \tag{9}$$

where p and q are algebraic polynomials of degree $\leqslant n$ and $\leqslant m$, respectively, has points of discontinuity (of course, for $m \geqslant 1$). In this section, we apply the above results to some classes of generalized rational functions (including $\mathscr{R}_{n,m}$) and obtain some results on radial continuity of the metric projection operator to various classes of generalized rational functions. (Note that the metric projection onto rational functions is in general discontinuous.)

Let Q be a metrizable compact set. Consider the class of generalized rational functions

$$\mathscr{R}_W^V := \big\{r = v/w \;\big|\; v \in V,\ w \in W\big\},$$

where $V, W \subset C(Q)$ are convex sets, W consists of positive functions. It is known that $\mathscr{R}_W^V$ is a strict protosun in $C(Q)$.

We also consider the following generalization of the above class $\mathscr{R}_W^V$. Let $V, W \subset C(Q)$, and let $U \subset V \times W$ be a nonempty convex set. We

set

$$\mathscr{R}_U := \big\{ r \in C(Q) \mid rw = v, \quad w \not\equiv 0, \quad (v,w) \in U \big\}. \tag{10}$$

Theorem 3.A. *The set of generalized rational functions $\mathscr{R}_U$ (see (10)) is a monotone path-connected strict protosun in $C(Q)$.*

Definition 3.1. Let Q be a Hausdorff compact set and let $U \subset (V \times W)$, where $V, W \subset C(Q)$. The set

$$\mathscr{R}_U := \big\{ r \in C(Q) \mid rw = v, \quad w \not\equiv 0, \quad (v,w) \in U \big\}$$

is said to be *algebraically complete* if the conditions: (a) there exists a sequence $(v_k, w_k) \to (v, w)$, in the space $C(Q) \times C(Q)$, where $(v_k, w_k) \in U$, $w \not\equiv 0$, and (b) there exists a function $r \in C(Q)$ such that $r(t) = v(t)/w(t)$ for all $t \in Q \setminus Z(w)$, where $Z(w)$ is the set of zeros of the function w, are equivalent to the condition $(v, w) \in U$.

In Theorem 3.B and below, Δ is the pointwise (Deutsch) convergence in $C(Q)$ on a dense subset of the compact set Q, *viz.*, $x_n \xrightarrow{\Delta} x$ if there exists a dense set $Q_0 \subset Q$, $\overline{Q_0} = Q$, such that $x_n(t) \to x(t)$ for each $t \in Q_0$ ($D = \Delta$); note that here Q_0 depends on (x_n).

Theorem 3.B. *Let $\mathscr{R}_U$ be algebraically complete and, for each nonzero function from W, the complement of its set of zeros in Q is dense in Q. Then $\mathscr{R}_U$ is boundedly Δ-compact in $C(Q)$ and, as a corollary, is proximinal in $C(Q)$.*

Corllary 3.A. *Let D be a compact domain in $\mathbb{R}^n$, $V, W \subset C(D)$ be nonempty boundedly compact sets composed of real-analytic functions, and let $U \subset V \times W$ be a nonempty set. Assume that $\mathscr{R}_U$ is algebraically complete. Then $\mathscr{R}_U$ is boundedly Δ-compact in $C(Q)$ and, as a corollary, is proximinal in $C(Q)$.*

We have the following result on radial continuity of the metric projection onto the class $\mathscr{R}_U$ of generalized rational fractions.

Theorem 3.1. *If $\mathscr{R}_U$ (see* (10)*) satisfies the conditions of Theorem* 3.B *or Corollary* 3.A*, then the metric projection onto $\mathscr{R}_U$ is* Δ-ORU*-continuous.*

Corollary 3.1. *The metric projection onto the set $\mathscr{R}_{n,m}$ (see* (9) *of classical rational functions in $C[a, b]$ is* ORL*-,* IRL*-, and* ORU*-continuous.*

Remark 3.1. It is well known that $\mathscr{R}_{n,m}$ is a Chebyshev sun in $C[a, b]$. However, $\mathscr{R}_U$ may fail to be an existence set or may have nonsingleton values of the metric projection operator.

4 Deutsch convergence

Definition 4.1. Let X be a normed linear space. By a *Deutsch convergence* (or a *D-convergence*) $x_\delta \overset{D}{\to} x$ of nets (sequences) in X we mean a convergence D with the following properties:

(i) D is norm dominated, i.e., $x_\delta \overset{D}{\to} x$ implies that $\|x\| \leqslant \limsup \|x_\delta\|$;

(ii) D is translation invariant, i.e., $x_\delta \overset{D}{\to} x$ implies that $x_\delta + y \overset{D}{\to} x + y$ for each $y \in X$;

(iii) D is homogeneous, i.e., $x_\delta \overset{D}{\to} x$ implies that $\alpha x_\delta \overset{D}{\to} \alpha x$ for each number α.

The above convergence is sequential if in Definition 4.1 we consider sequences rather than nets. It is clear that a Deutsch convergence is also a sequential Deutsch convergence, but not vice versa. Let us give some examples of Deutsch convergences.

Example 4.1. a) *Convergence in norm*:: $x_\delta \overset{n}{\to} x$ if $\|x_\delta - x\| \to 0$ (here $D = n$).

b) *Weak convergence* $x_\delta \overset{w}{\to} x$ if $x^*(x_\delta) \to x^*(x)$ for each $x^* \in X^*$ $(D = w)$.

c) $*$*-weak convergence* in $X = Y^*$: $y^*_\delta \overset{w^*}{\to} y^*$ if $y^*_\delta(y) \to y^*(y)$ for each $y \in Y$ $(D = w^*)$.

d) *Pointwise convergence in $C(Q)$ on a dense subset of the compact set Q*: $x_\delta \xrightarrow{\Delta} x$ if there exists a dense set $Q_0 \subset Q$, $\overline{Q_0} = Q$, such that $x_\delta(t) \to x(t)$ for each $t \in Q_0$ ($D = \Delta$); note that here Q_0 depends on (x_δ).

e) *Pointwise convergence in $C[a,b]$ at all except possibly finitely many points*: $x_\delta \xrightarrow{\varphi} x$ if there exists a set $T_0 \subset [a,b]$ such that $[a,b] \setminus T_0$ is finite and $x_\delta(t) \to x(t)$ for each $t \in T_0$.

f) In $L^p(\mu)$, the *convergence of sequences almost everywhere*: $x_n \xrightarrow{\text{ae}} x$ if $\mu(\{t \mid x_n(t) \not\to x(t)\}) = 0$ ($D = \text{ae}$). This is a sequential Deutsch convergence, but not a convergence for nets.

g) *Almost weak convergence*: $x_n \xrightarrow{\text{aw}} x$ if there exists a w^*-dense subset Λ of the set of extreme points of the unit sphere S^* of the dual space such that $x^*(x_\delta) \to x^*(x)$ for each $x^* \in \Lambda$ ($D = \text{aw}$). In $C(Q)$, aw-convergence coincides with the Δ-convergence defined in d).

h) Let Σ is a σ-algebra on Ω, μ be a σ-finite measure on Σ. We say that a sequence of functions (x_n) (where $x_n : \Omega \to \mathbb{R}$) aes-*converges*[1] to a function $x : \Omega \to \mathbb{R}$ if, for any set $A \in \Sigma$, $\mu(A) < \infty$, there exists a subsequence of numbers (n_k) such that (x_{n_k}) converges to x almost everywhere on A.

We now define D-open, D-closed, and D-compact sets.

[1]The "aes" stands for "almost everywhere convergence of sequence".

Definition 4.2. Let D be a Deutsch convergence (a Deutsch sequential convergence). A subset M of a space X is called *D-closed* if M contains the limit of each of its D-convergent nets (sequences). A set M is *D-open* if $X \setminus M$ is D-closed.

A set M is (countably) *D-compact* if each sequence from M contains a subsequence which D-converges to a point from M.

Definition 4.3. Let D be a Deutsch convergence on X, $\varnothing \neq M \subset X$ and $x_0 \in X$. The metric projection P_M is called D-ORU-*continuous* (*outer radially upper Deutsch continuous*) at a point x_0 if, for each $y_0 \in P_M x_0$ and each D-open $W \supset P_M x_0$, there exists a neighborhood U of x_0 such that $P_M x \subset W$ for each $x \in U \cap \{y_0 + \lambda(x_0 - y_0) \mid \lambda \geqslant 1\}$.

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