SIAM J. CONTROL
Vol. 12, No. 1, February 1974

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FINDING THE POINT OF A POLYHEDRON CLOSEST TO THE ORIGIN*

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Abstract. An algorithm is given for finding the point of a convex polyhedron in an n-dimensional Euclidean space which is closest to the origin. It is assumed that the convex polyhedron is defined as the convex hull of a given finite set of points. This problem arises when one wishes to determine the direction of steepest descent for certain minimax problems.

1. Let a finite set of points $H = \{z_i\}_{i=1}^{s}$ be given in an n-dimensional Euclidean space E_n. We denote by L the convex hull of the points z_i:

$$L = \left\{ z = \sum_{i=1}^{s} \alpha_i z_i \middle| \alpha_i \geqq 0, \sum_{i=1}^{s} \alpha_i = 1 \right\}.$$

Obviously, L is a bounded closed convex set. We shall denote by z^* the point of L which is closest to the origin

$$(z^*, z^*) = \min_{z \in L} (z, z).$$

Our goal is to describe a new method of successive approximations for finding the point z^*.

2. It is not difficult to show that the point z^* exists and is unique. Moreover, the following inequality holds for any $z \in L$ (see, e.g., [1]):

$$(z, z^*) \geqq (z^*, z^*). \tag{1}$$

We set

$$\delta(z) = (z, z) - \min_{i \in [1:s]} (z_i, z).$$

Since

$$(v, z) \geqq \min_{i \in [1:s]} (z_i, z) \tag{2}$$

for any $v, z \in L$, we have $\delta(z) \geqq 0$ if $z \in L$.

The following lemma also follows immediately from (1) and (2).

LEMMA 1. *The inequality*

$$\|z - z^*\| \leqq \min \{\sqrt{\delta(z)}, \|z\|\} \tag{3}$$

holds for any $z \in L$.

COROLLARY 1. *If a sequence of points* $v_k \in L$, $k = 0, 1, 2, \cdots$, *is such that* $\delta(v_k) \xrightarrow[k \to \infty]{} 0$, *then*

$$v_k \xrightarrow[k \to \infty]{} z^*.$$

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* Originally published in *Vestnik Leningrad Univ.*, 19 (1971), pp. 38–45. Submitted December 30, 1969. This translation into English has been prepared by K. Makowski.

COROLLARY 2. *If a sequence of points* $v_k \in L$, $k = 0, 1, 2, \cdots$, *is such that* $\|v_{k+1}\| \leqq \|v_k\|$, *and if there exists a subsequence* $\{v_{k_j}\}$ *for which* $\delta(v_{k_j}) \xrightarrow[j\to\infty]{} 0$, *then* $v_k \xrightarrow[k\to\infty]{} z^*$.

The following theorem holds.

THEOREM 1. *For a point* $\bar{z} \in L$ *to be the point of* L *closest to the origin, it is necessary and sufficient that* $\delta(\bar{z}) = 0$.

Proof. The *sufficiency* follows from (3). The *necessity*. Let $\bar{z} = z^*$. Then, first, $\delta(z^*) \geqq 0$. On the other hand, we have by virtue of (1), $(z_i, z^*) \geqq (z^*, z^*)$ for any $i \in [1:s]$, because $z_i \in H \subset L$. Hence, $\min_{i\in[1:s]}(z_i, z^*) \geqq (z^*, z^*)$ or, which is the same, $\delta(z^*) \leqq 0$. Therefore, $\delta(z^*) = 0$. The theorem has been proved.

3. We denote by Ξ the set of vectors A of the form

$$A = (\alpha_1, \cdots, \alpha_s), \qquad \alpha_i \geqq 0, \quad \sum_{i=1}^{s} \alpha_i = 1.$$

We set

$$z(A) = \sum_{i=1}^{s} \alpha_i z_i;$$

(4)

$$\Delta(A) = \max_{\{i|\alpha_i > 0\}} (z_i, z(A)) - \min_{i\in[1:s]} (z_i, z(A)).$$

We denote by $i' = i'(A)$ a subscript at which the maximum in the right-hand side of (4) is attained (if there are several such subscripts, then we take any one of them). Thus, $\alpha_{i'} > 0$ and $(z_{i'}, z(A)) = \max_{\{i|\alpha_i > 0\}} (z_i, z(A))$.

LEMMA 2. *The inequalities* $\alpha_{i'}\Delta(A) \leqq \delta(v) \leqq \Delta(A)$ *hold for any vector* $v = z(A)$, $A \in \Xi$.

Proof. We note that

$$(v, v) = \sum_{i=1}^{s} \alpha_i(z_i, z(A)) \leqq \max_{\{i|\alpha_i > 0\}} (z_i, z(A)).$$

Hence, the inequality $\delta(v) \leqq \Delta(A)$ follows. We denote by $z_{i''}$, $i'' = i''(A)$, the point of the set H for which

$$(z_{i''}, z(A)) = \min_{i\in[1:s]} (z_i, z(A)).$$

In this case

(5) $$\Delta(A) = (z_{i'} - z_{i''}, z(A)).$$

We set $\bar{A} = \{\bar{\alpha}_1, \cdots, \bar{\alpha}_s\} \in \Xi$, where

$$\bar{\alpha}_i = \begin{cases} \alpha_i & \text{for } i \neq i', i'', \\ 0 & \text{for } i = i', \\ \alpha_{i'} + \alpha_{i''} & \text{for } i = i''. \end{cases}$$

Obviously,

(6) $$z(\bar{A}) = z(A) + \alpha_{i'}(z_{i''} - z_{i'}).$$

Since $z(\bar{A}) \in L$, we have by virtue of (2)

(7) $$(z(\bar{A}), z(A)) \geqq \min_{i\in[1:s]} (z_i, z(A)).$$

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Taking into account (7), (6), and (5), we obtain $\delta(z(A)) \geqq \alpha_{i'}\Delta(A)$. The lemma has been proved.

THEOREM 2. *For a point $v = z(A)$, $A \in \Xi$, to be the point of L closest to the origin, it is necessary and sufficient that $\Delta(A) = 0$.*

The proof follows in an obvious way from Lemma 2 and Theorem 1.

4. We shall now describe the method of successive approximations for finding the point z^*. We choose a vector $A_0 \in \Xi$ in an arbitrary way, and we set $v_0 = z(A_0)$. Assume that the kth approximation $v_k \in L : v_k = z(A_k)$, $A_k = (\alpha_1^{(k)}, \cdots, \alpha_s^{(k)}) \in \Xi$, has already been found. We describe the construction of v_{k+1}.

First of all, we find vectors $z_{i_k'}$ and $z_{i_k''}$ of H such that

$$(z_{i_k'}, v_k) = \max_{\{i \mid \alpha_i^{(k)} > 0\}} (z_i, z(A_k)),$$

$$(z_{i_k''}, v_k) = \min_{i \in [1:s]} (z_i, z(A_k)).$$

In this case,

$$\Delta_k \stackrel{\text{def}}{=} \Delta(A_k) = (z_{i_k'} - z_{i_k''}, v_k). \tag{8}$$

We consider the interval

$$\dot{v}_k(t) = v_k + t\alpha_{i_k'}^{(k)}(z_{i_k''} - z_{i_k'}), \qquad 0 \leqq t \leqq 1. \tag{9}$$

Let t_k with $0 \leqq t_k \leqq 1$ be determined by the relation

$$(v_k(t_k), v_k(t_k)) = \min_{0 \leqq t \leqq 1} (v_k(t), v_k(t)).$$

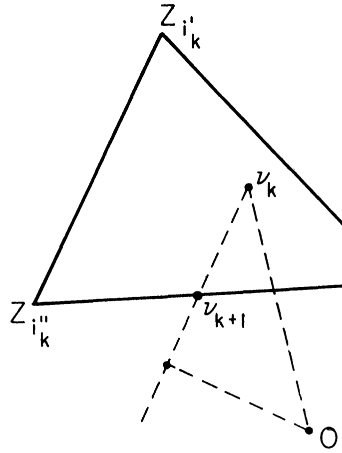

FIG. 1

We set $v_{k+1} = v_k(t_k)$ (see Fig. 1). It is not difficult to verify that $v_{k+1} = z(A_{k+1})$, where

$$A_{k+1} = (\alpha_1^{(k+1)}, \cdots, \alpha_s^{(k+1)}) \in \Xi,$$

$$\alpha_i^{(k+1)} = \begin{cases} \alpha_i^{(k)} & \text{for } i \neq i'_k, i''_k, \\ a_{i'_k}^{(k)} - t_k \alpha_{i'_k}^{(k)} & \text{for } i = i'_k, \\ \alpha_{i''_k}^{(k)} + t_k \alpha_{i'_k}^{(k)} & \text{for } i = i''_k. \end{cases}$$

For the sake of simplicity, we shall subsequently make use of the following notation: $\alpha'_k = \alpha_{i'_k}^{(k)}$, $z'_k = z_{i'_k}$, $\bar{z}_k = z_{i''_k}$.

Continuing the process described, we obtain the sequence of points $v_k \in L$, $k = 0, 1, 2, \cdots$, with

(10) $$\|v_{k+1}\| \leqq \|v_k\|.$$

LEMMA 3. *The following limit relation holds*:

(11) $$\lim_{k \to \infty} \alpha'_k \Delta_k = 0.$$

Proof. First, we note that by virtue of (8) and (9),

(12) $$(v_k(t), v_k(t)) = (v_k, v_k) - 2t\alpha'_k\Delta_k + t^2(\alpha'_k\|\bar{z}_k - z'_k\|)^2.$$

Assume that the assertion of the lemma is false. Then there exists a subsequence $\{v_{k_j}\}$ for which $\alpha'_{k_j}\Delta_{k_j} \geqq \varepsilon > 0$. By virtue of (12) we have, for all $t \in [0, 1]$ and uniformly with respect to k_j,

$$(v_{k_j}(t), v_{k_j}(t)) \leqq (v_{k_j}, v_{k_j}) - 2t\varepsilon + t^2 d^2,$$

where $d = \max_{l,p \in [1:s]} \|z_l - z_p\| > 0$. Hence, it follows that the following inequality holds for $t_0 = \min\{\varepsilon/d^2, 1\}$ (obviously, $0 < t_0 \leqq 1$):

$$(v_{k_j}(t_0), v_{k_j}(t_0)) \leqq (v_{k_j}, v_{k_j}) - t_0\varepsilon.$$

Taking into account that, by definition, $(v_{k_j+1}, v_{k_j+1}) \leqq (v_{k_j}(t_0), v_{k_j}(t_0))$, we obtain

$$(v_{k_j+1}, v_{k_j+1}) \leqq (v_{k_j}, v_{k_j}) - t_0\varepsilon$$

uniformly with respect to k_j.

The number of such reductions in the monotonically nonincreasing sequence (v_k, v_k) is infinite, which contradicts the fact that all the (v_k, v_k) are nonnegative. The lemma has been proved.

LEMMA 4. *The limit relation*

(13) $$\varliminf_{k \to \infty} \Delta_k = 0$$

holds.

Proof. Assume the contrary: $\varliminf_{k \to \infty} \Delta_k = \Delta' > 0$. Then we have

(14) $$\Delta_k \geqq \Delta'/2$$

for numbers $k \geqq k_0$ sufficiently large. Taking into account (11), we conclude that

(15) $$\alpha'_k \xrightarrow[k \to \infty]{} 0.$$

We also note that, by virtue of (12) and (14),

(16) $$\|v_{k+1}\| < \|v_k\|$$

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for $k \geqq k_0$. We denote by $\bar{t}_k$ the point at which $(v_k(t), v_k(t))$ attains its global minimum. Obviously (see (12)),

$$\bar{t}_k = \frac{\Delta_k}{\alpha'_k \|\bar{z}_k - z'_k\|^2}.$$

By virtue of (14) and (15), $\bar{t}_k \xrightarrow[k \to \infty]{} \infty$. Hence, it follows that for numbers $k \geqq k_1 \geqq k_0$ sufficiently large, the minimum of $(v_k(t), v_k(t))$ on the interval $0 \leqq t \leqq 1$ is attained for $t_k = 1$. Therefore, for these k,

$$v_{k+1} = v_k + \alpha'_k(\bar{z}_k - z'_k). \tag{17}$$

However, the sequence of points $v_{k_1}, v_{k_1+1}, v_{k_1+2}, \cdots$, which are connected by relation (17), can contain only a finite number of mutually distinct elements,[1] which contradicts (16). The lemma has been proved.

THEOREM 3. *The sequence $\{v_k\}$ constructed above converges to the point z^*.*

The proof follows from Lemmas 4 and 2 and from Corollary 2 to Lemma 1 in an obvious way.

Remark. If it turns out that, for some k, $\Delta_k = 0$, i.e., that $v_k = z^*$, then $v_{k+1} = z^*$ for all $j = 1, 2, \cdots$. This fact follows from (12).

5. We note certain peculiarities of the method of successive approximations described in the preceding section. We introduce the hyperplane

$$G = \{z | (z, z^*) = (z^*, z^*)\}.$$

THEOREM 4. *If $z^* \neq \mathbf{0}$, i.e., if the origin does not belong to L, then, beginning with some number, $v_k \in G$.*

Proof. We note that, by virtue of Theorem 1,

$$\min_{i \in [1:s]} (z_i, z^*) = (z^*, z^*).$$

We set

$$H_1 = \{z_i \in H | (z_i, z^*) = (z^*, z^*)\}; \quad H_2 = H \setminus H_1 = \{z_i \in H | (z_i, z^*) > (z^*, z^*)\}.$$

If H_2 is an empty set, then $v_k \in G$ for all $k = 0, 1, 2, \cdots$. Therefore, we henceforth assume that H_2 is a nonempty set. We introduce the notation

$$\tau = \min_{z_i \in H_2} (z_i, z^*) - (z^*, z^*) > 0.$$

Since $v_k \xrightarrow[k \to \infty]{} z^*$, we have

$$\max_{i \in [1:s]} |(z_i, v_k) - (z_i, z^*)| < \tau/4$$

for numbers $k \geqq k_0$ sufficiently large. It is not difficult to show that the following relations hold for the same numbers $k \geqq k_0$:

$$(z_i, v_k) \leqq (z^*, z^*) + \tau/4 \quad \text{if } z_i \in H_1;$$

$$(z_i, v_k) \geqq (z^*, z^*) + 3\tau/4 \quad \text{if } z_i \in H_2.$$

[1] This remark is due to M. S. Al'tmark.

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Hence, it follows that

(18) $$\min_{i\in[1:s]} (z_i, v_k) = \min_{z_i\in H_1} (z_i, v_k).$$

Further, if a point $z_i \in H_2$ enters the representation of v_k, $k \geqq k_0$, with a nonzero coefficient, then

(19) $$\Delta_k \geqq \tau/2,$$

where

(20) $$\|v_{k+1}\| < \|v_k\|.$$

Let

$$v_k = \sum_{\{i|z_i\in H_1\}} \alpha_i^{(k)} z_i + \sum_{\{i|z_i\in H_2\}} \alpha_i^{(k)} z_i.$$

By virtue of the definitions of H_1 and τ,

$$(v_k - z^*, z^*) = \sum_{\{i|z_i\in H_2\}} \alpha_i^{(k)}(z_i - z^*, z^*) \geqq \tau \sum_{\{i|z_i\in H_2\}} \alpha_i^{(k)}.$$

Since the left-hand side of this inequality tends to zero as $k \to \infty$,

$$\sum_{\{i|z_i\in H_2\}} \alpha_i^{(k)} \xrightarrow[k\to\infty]{} 0.$$

We choose a large $k_1 \geqq k_0$ such that the following inequality holds for $k \geqq k_1$:

(21) $$\sum_{\{i|z_i\in H_2\}} \alpha_i^{(k)} \leqq \frac{\tau}{2d^2},$$

where $d = \max_{z_i\in H_1, z_j\in H_2} \|z_i - z_j\| > 0$. We denote by $\bar{t}_k$ the point at which $(v_k(t), v_k(t))$ attains its global minimum. If $z_i \in H_2$ enters the representation of v_k, $k \geqq k_1$, with a nonzero coefficient, then we obtain by virtue of (19) and (21)

$$\bar{t}_k = \frac{\Delta_k}{\alpha_k'\|\bar{z}_k - z_k'\|^2} \geqq \frac{\tau}{2\left(\sum_{\{i|z_i\in H_2\}} \alpha_i^{(k)}\right) d^2} \geqq 1.$$

Hence, it follows that

(22) $$v_{k+1} = v_k + \alpha_k'(\bar{z}_k - z_k').$$

We assume that points $z_i \in H_2$ enter the representations of all vectors

(23) $$v_{k_1}, v_{k_1+1}, v_{k_1+2}, \cdots$$

with nonzero coefficients. By virtue of (22), sequence (23) contains only a finite number of mutually distinct elements. However, this contradicts (20). Therefore, there exists a point $v_{\bar{k}}$, $\bar{k} \geqq k_1$, which has the following representation:

$$v_{\bar{k}} = \sum_{\{i|z_i\in H_1\}} \alpha_i^{(\bar{k})} z_i; \quad \alpha_i^{(\bar{k})} \geqq 0, \quad \sum_{\{i|z_i\in H_1\}} \alpha_i^{(\bar{k})} = 1.$$

By virtue of (18), all the v_k with $k \geqq \bar{k}$ have similar representations. In particular, we have by definition of H_1, for $k \geqq \bar{k}$, that $(v_k, z^*) = (z^*, z^*)$, i.e., $v_k \in G$. The theorem has been proved.

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THEOREM 5. *The limit relation*

$$\lim_{k\to\infty} \Delta_k = 0$$

holds.

Proof. If $z^* = \mathbf{0}$, then the assertion of the theorem follows from the definition of Δ_k and from the fact that $\|v_k\| \xrightarrow[k\to\infty]{} 0$. Therefore, we assume that $z^* \neq \mathbf{0}$. By virtue of Theorem 4, we have for $k \geqq \bar{k}$,

$$\Delta_k \leqq \max_{\{i|z_i\in H_1\}} (z_i, v_k) - \min_{\{i|z_i\in H_1\}} (z_i, v_k).$$

According to the definition of H_1, the right-hand side of this inequality tends to zero as $k \to \infty$. Therefore, $\Delta_k \xrightarrow[k\to\infty]{} 0$ also, since $\Delta_k \geqq 0$. The theorem has been proved.

6. We set

$$\tilde{\Delta}_k = \Delta_k/\|v_k\|^2, \qquad k = 0, 1, 2, \cdots.$$

If $v_k = \mathbf{0}$, then we set by definition $\tilde{\Delta}_k = \infty$.

THEOREM 6. *For the origin to belong to the set L, it is necessary and sufficient that the following inequality hold for all $k = 0, 1, 2, \cdots$:*

$$\tilde{\Delta}_k \geqq 1. \tag{24}$$

Proof. The necessity. We have $z^* = \mathbf{0}$. Taking into account that $v_k \in L$, we obtain on the basis of Lemmas 1 and 2

$$\tilde{\Delta}_k = \frac{\Delta_k}{\|v_k\|^2} \geqq \frac{\delta(v_k)}{\|v_k\|^2} \geqq 1.$$

The sufficiency. Assume that $z^* \neq \mathbf{0}$. Then $\tilde{\Delta}_k \leqq \Delta_k/\|z^*\|^2$. By virtue of Theorem 5, we obtain $\tilde{\Delta}_k \xrightarrow[k\to\infty]{} 0$, which contradicts (24). The theorem has been proved.

Thus, if $z^* = \mathbf{0}$, then $\|v_k\| \xrightarrow[k\to\infty]{} 0$ and, for all $k = 0, 1, 2, \cdots$, the inequality $\tilde{\Delta}_k \geqq 1$ holds. If $z^* \neq \mathbf{0}$, then $\|v_k\| \geqq \|z^*\|$ and $\tilde{\Delta}_k \xrightarrow[k\to\infty]{} 0$.

If the inequality $\tilde{\Delta}_k < 1$ holds for some k, then, by virtue of Theorem 6, the origin does not belong to the set L. Moreover, it is not difficult to prove that, in this case, the hyperplane $(v_k, z) - (v_k, \bar{z}_k) = 0$ strictly separates the origin from L.

7. We remind the reader that $v_k = z(A_k)$, $A_k \in \Xi$. We set $I_k = \{i|\alpha_i^{(k)} > 0\}$ and introduce the set

$$B_k = \left\{ z = z_{i_0} + \sum_{\substack{i\in I_k\\ i\neq i_0}} \alpha_i(z_i - z_{i_0}) | \alpha_i \in (-\infty, \infty) \right\}.$$

Here, i_0 is an arbitrary subscript of I_k. We denote by $\tilde{v}_k$ the vector of B_k with the smallest norm: $\|\tilde{v}_k\| = \min_{z\in B_k} \|z\|$. We note that the point $\tilde{v}_k \in B_k$ is unique, although its representation in the form

$$\tilde{v}_k = z_{i_0} + \sum_{\substack{i\in I_k\\ i\neq i_0}} \bar{\alpha}_i(z_i - z_{i_0})$$

may not be unique.

It is not difficult to show that the numbers $\bar{\alpha}_l$ constitute the solution of the following linear system:

$$\left(z_{i_0} + \sum_{\substack{i \in I_k \\ i \neq i_0}} \alpha_i(z_i - z_{i_0}), z_j - z_{i_0}\right) = 0, \qquad j \in I_k, \quad j \neq i_0. \tag{25}$$

THEOREM 7. *There exists an infinite subsequence of vectors* $\{\tilde{v}_{k_j}\}$ *such that* $\tilde{v}_{k_j} = z^*$ *for all* k_j.

Proof. We shall assume that $z^* \neq \mathbf{0}$ (if $z^* = \mathbf{0}$, then the proof is only simplified). First, we separate a subsequence $\{v_{k_j}\}$ such that

(i) $\alpha_i^{(k_j)} \xrightarrow[j \to \infty]{} \alpha_i^*$, $i \in [1:s]$, in this case,

$$v_{k_j} \xrightarrow[j \to \infty]{} z^* = \sum_{i=1}^{s} \alpha_i^* z_i;$$

(ii) $v_{k_j} \in G$ (see Theorem 4).

We set $I^* = \{i | \alpha_i^* > 0\}$. Obviously, we have for k_j sufficiently large

$$I^* \subset I_{k_j}. \tag{26}$$

Henceforth, we consider only such k_j. We denote by L^* the convex hull of the points z_i, $i \in I^*$. Obviously, $z^* \in L^*$. We denote by L_{k_j} the convex hull of the points z_i, $i \in I_{k_j}$. By virtue of (ii), (26), and of the definition of the set B_{k_j}, we have

$$L^* \subset L_{k_j} \subset B_{k_j} \subset G.$$

Further, $\|z^*\| = \min_{z \in G} \|z\| \leqq \min_{z \in B_{k_j}} \|z\|$. Since $z^* \in L^*$, $z^* \in B_{k_j}$. Therefore, $\|z^*\| = \min_{z \in B_{k_j}} \|z\|$.

Taking into account that the point of B_{k_j} with the smallest norm is unique, we obtain $\tilde{v}_{k_j} = z^*$. The theorem has been proved.

On the basis of this theorem, one can assert that finding z^* reduces to solving a finite number of systems of linear equations of form (25). We note that it is purposeful to solve these systems only for the k for which either $\|v_k\|$ or $\tilde{\Delta}_k$ is sufficiently small.

Regarding other methods of finding the point of a polyhedron which is closest to the origin, see [2]–[4].

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